

# A construction of Hom-Yetter-Drinfeld category

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## ABSTRACT

In continuation of our recent work about smash product Hom-Hopf algebras in [11], we introduce Hom-Yetter-Drinfeld category  ${}^H_H\mathbb{YD}$  via Radford biproduct Hom-Hopf algebra, and prove that the Hom-Yetter-Drinfeld modules can provide solutions of the Hom-Yang-Baxter equation and  ${}^H_H\mathbb{YD}$  is a pre-braided tensor category, where  $(H, \beta, S)$  is a Hom-Hopf algebra. Furthermore, we obtain that  $(A \bowtie H, \alpha \otimes \beta)$  is a Radford biproduct Hom-Hopf algebra if and only if  $(A, \alpha)$  is a Hopf algebra in the category  ${}^H_H\mathbb{YD}$ . At last, some examples and applications are given.

**Key words:** Hom-smash (co)product; Hom-Yetter-Drinfeld category; Radford biproduct; Hom-Yang-Baxter equation.

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## 1 Introduction

The motivation to introduce Hom-type algebras comes for examples related to  $q$ -deformations of Witt and Virasoro algebras, which play an important role in physics, mainly in conformal field theory. Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) have been intensively investigated in the literature recently, see [2, 3, 5, 8–11, 16–19, 24–32]. Hom-algebras are generalizations of algebras obtained by a twisting map, which have been introduced for the first time in [18] by Makhlouf and Silvestrov. The associativity is replaced by Hom-associativity, Hom-coassociativity for a Hom-coalgebra can be considered in a similar way.

In [24, 29], Yau introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras and the dual version (i.e. comodule Hom-coalgebras) was studied by Zhang in [31]. Based on Yau's definition of module Hom-algebras, Ma-Li-Yang in [11] constructed smash product Hom-Hopf algebra  $(A \bowtie H, \alpha \otimes \beta)$

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generalizing the Molnar's smash product (see [13]), and gave the cobraided structure (in the sense of Yau's definition in [28]) on  $(A \sharp H, \alpha \otimes \beta)$ , and also considered the case of twist tensor product Hom-Hopf algebra. Makhlouf and Panaite defined and studied a class of Yetter-Drinfeld modules over Hom-bialgebras in [16] and derived the constructions of twistors, pseudotwistors, twisted tensor product and smash product in the setting of Hom-case in [17].

Yetter-Drinfeld modules are known to be at the origin of a very vast family of solutions to the Yang-Baxter equation. Let  $H$  be a bialgebra,  $A$  a left  $H$ -module algebra and a left  $H$ -comodule coalgebra. In [20], Radford gave a construction of bialgebra (called Radford biproduct bialgebra) by combining the smash product algebra  $A \# H$  with the smash coproduct coalgebra  $A \times H$ . Majid (see [14, 15]) made the following conclusion:  $A$  is a bialgebra in Yetter-Drinfeld category  ${}^H_H\mathcal{YD}$  if and only if  $A \star H$  is a Radford biproduct. The Radford biproduct plays an important role in the lifting method for the classification of finite dimensional pointed Hopf algebras (see [1]).

In this paper, we introduce Hom-Yetter-Drinfeld category  ${}^H_H\mathcal{YD}$  via Radford biproduct Hom-Hopf algebra, and prove that the Hom-Yetter-Drinfeld modules can provide solutions of the Hom-Yang-Baxter equation. Furthermore, we obtain that  $(A \sharp H, \alpha \otimes \beta)$  is a Radford biproduct Hom-Hopf algebra if and only if  $(A, \alpha)$  is a Hom-Hopf algebra in the category  ${}^H_H\mathcal{YD}$ .

This article is organized as follows. In Section 2, we recall some definitions and results which will be used later. Let  $(H, \beta)$  be a Hom-bialgebra,  $(A, \alpha)$  a left  $(H, \beta)$ -module Hom-algebra and a left  $(H, \beta)$ -comodule Hom-coalgebra. In [11], the smash product Hom-algebra  $(A \sharp H, \alpha \otimes \beta)$  was constructed. In Section 3, we first define smash coproduct Hom-coalgebra  $(A \diamond H, \alpha \otimes \beta)$  (see Proposition 3.1), then derive necessary and sufficient conditions for  $(A \sharp H, \alpha \otimes \beta)$  and  $(A \diamond H, \alpha \otimes \beta)$  to be a Hom-bialgebra, which is called Radford biproduct Hom-bialgebra and denoted by  $(A \sharp H, \alpha \otimes \beta)$  (see Theorem 3.3, 3.6). In Section 4, we introduce the concept of Hom-Yetter-Drinfeld category  ${}^H_H\mathcal{YD}$  (see Definition 4.1, 4.2), which is different from the one defined by Makhlouf and Panaite in [16], the one defined by Chen and Zhang in [5] and the one defined by Liu and Shen in [9]. We also prove that the Hom-Yetter-Drinfeld modules can provide solutions of the Hom-Yang-Baxter equation in the sense of Yau's definition in [26, 27, 30] (see Proposition 4.3) and that  ${}^H_H\mathcal{YD}$  is a pre-braided tensor category (see Theorem 4.7). Furthermore, we obtain that  $(A \sharp H, \alpha \otimes \beta)$  is a Radford biproduct Hom-Hopf algebra if and only if  $(A, \alpha)$  is a Hom-Hopf algebra in the category  ${}^H_H\mathcal{YD}$  (see Theorem 4.8), which generalizes the Majid's result in [14, 15]. In last section, some examples and applications are given.

Throughout this paper we freely use the Hopf algebras and coalgebras terminology introduced in [6, 21–23].

The authors were informed by the Editor that the following paper [4] related with the

subject of our paper is accepted for publication.

## 2 Preliminaries

Throughout this paper, we follow the definitions and terminologies in [7, 11, 24, 26, 31], with all algebraic systems supposed to be over the field  $K$ . Given a  $K$ -space  $M$ , we write  $id_M$  for the identity map on  $M$ .

We now recall some useful definitions.

**Definition 2.1** A Hom-algebra is a quadruple  $(A, \mu, 1_A, \alpha)$  (abbr.  $(A, \alpha)$ ), where  $A$  is a  $K$ -linear space,  $\mu : A \otimes A \longrightarrow A$  is a  $K$ -linear map,  $1_A \in A$  and  $\alpha$  is an automorphism of  $A$ , such that

$$\begin{aligned} (HA1) \quad & \alpha(aa') = \alpha(a)\alpha(a'); \quad \alpha(1_A) = 1_A, \\ (HA2) \quad & \alpha(a)(a'a'') = (aa')\alpha(a''); \quad a1_A = 1_Aa = \alpha(a) \end{aligned}$$

are satisfied for  $a, a', a'' \in A$ . Here we use the notation  $\mu(a \otimes a') = aa'$ .

Let  $(A, \alpha)$  and  $(B, \beta)$  be two Hom-algebras. Then  $(A \otimes B, \alpha \otimes \beta)$  is a Hom-algebra (called tensor product Hom-algebra) with the multiplication  $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$  and unit  $1_A \otimes 1_B$ .

**Definition 2.2** A Hom-coalgebra is a quadruple  $(C, \Delta, \varepsilon_C, \beta)$  (abbr.  $(C, \beta)$ ), where  $C$  is a  $K$ -linear space,  $\Delta : C \longrightarrow C \otimes C$ ,  $\varepsilon_C : C \longrightarrow K$  are  $K$ -linear maps, and  $\beta$  is an automorphism of  $C$ , such that

$$\begin{aligned} (HC1) \quad & \beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2); \quad \varepsilon_C \circ \beta = \varepsilon_C \\ (HC2) \quad & \beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2); \quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c) \end{aligned}$$

are satisfied for  $c \in A$ . Here we use the notation  $\Delta(c) = c_1 \otimes c_2$  (summation implicitly understood).

Let  $(C, \alpha)$  and  $(D, \beta)$  be two Hom-coalgebras. Then  $(C \otimes D, \alpha \otimes \beta)$  is a Hom-coalgebra (called tensor product Hom-coalgebra) with the comultiplication  $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$  and counit  $\varepsilon_C \otimes \varepsilon_D$ .

**Definition 2.3** A Hom-bialgebra is a sextuple  $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$  (abbr.  $(H, \gamma)$ ), where  $(H, \mu, 1_H, \gamma)$  is a Hom-algebra and  $(H, \Delta, \varepsilon, \gamma)$  is a Hom-coalgebra, such that  $\Delta$  and  $\varepsilon$  are morphisms of Hom-algebras, i.e.

$$\begin{aligned} \Delta(hh') &= \Delta(h)\Delta(h'); \quad \Delta(1_H) = 1_H \otimes 1_H, \\ \varepsilon(hh') &= \varepsilon(h)\varepsilon(h'); \quad \varepsilon(1_H) = 1. \end{aligned}$$

Furthermore, if there exists a linear map  $S : H \longrightarrow H$  such that

$$S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H \text{ and } S(\gamma(h)) = \gamma(S(h)),$$

then we call  $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$  (abbr.  $(H, \gamma, S)$ ) a Hom-Hopf algebra.

Let  $(H, \gamma)$  and  $(H', \gamma')$  be two Hom-bialgebras. The linear map  $f : H \rightarrow H'$  is called a Hom-bialgebra map if  $f \circ \gamma = \gamma' \circ f$  and at the same time  $f$  is a bialgebra map in the usual sense.

**Definition 2.4** (see [24, 29]) Let  $(A, \beta)$  be a Hom-algebra. A left  $(A, \beta)$ -Hom-module is a triple  $(M, \triangleright, \alpha)$ , where  $M$  is a linear space,  $\triangleright : A \otimes M \rightarrow M$  is a linear map, and  $\alpha$  is an automorphism of  $M$ , such that

$$\begin{aligned} (HM1) \quad & \alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m), \\ (HM2) \quad & \beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m); \quad 1_A \triangleright m = \alpha(m) \end{aligned}$$

are satisfied for  $a, a' \in A$  and  $m \in M$ .

Let  $(M, \triangleright_M, \alpha_M)$  and  $(N, \triangleright_N, \alpha_N)$  be two left  $(A, \beta)$ -Hom-modules. Then a linear morphism  $f : M \rightarrow N$  is called a morphism of left  $(A, \beta)$ -Hom-modules if  $f(h \triangleright_M m) = h \triangleright_N f(m)$  and  $\alpha_M \circ f = f \circ \alpha_N$ .

**Remarks** (1) It is obvious that  $(A, \mu, \beta)$  is a left  $(A, \beta)$ -Hom-module.

(2) When  $\beta = id_A$  and  $\alpha = id_M$ , a left  $(A, \beta)$ -Hom-module is the usual left  $A$ -module.

**Definition 2.5** (see [24, 29]) Let  $(H, \beta)$  be a Hom-bialgebra and  $(A, \alpha)$  a Hom-algebra. If  $(A, \triangleright, \alpha)$  is a left  $(H, \beta)$ -Hom-module and for all  $h \in H$  and  $a, a' \in A$ ,

$$\begin{aligned} (HMA1) \quad & \beta^2(h) \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a'), \\ (HMA2) \quad & h \triangleright 1_A = \varepsilon_H(h)1_A, \end{aligned}$$

then  $(A, \triangleright, \alpha)$  is called an  $(H, \beta)$ -module Hom-algebra.

**Remarks** (1) When  $\alpha = id_A$  and  $\beta = id_H$ , an  $(H, \beta)$ -module Hom-algebra is the usual  $H$ -module algebra.

(2) Similar to the case of Hopf algebras, in [24, 29], Yau concluded that the Eq.(HMA1) is satisfied if and only if  $\mu_A$  is a morphism of  $H$ -modules for suitable  $H$ -module structures on  $A \otimes A$  and  $A$ , respectively.

(3) The smash product Hom-Hopf algebra  $(A \sharp H, \alpha \otimes \beta)$  is different from the one defined by Chen, Wang and Zhang in [3], since here the construction of  $(A \sharp B, \alpha \otimes \beta)$  is based on the concept of the module Hom-algebra introduced by Yau in [24, 29], while two of conditions (6.1), (6.2) in the module Hom-algebra in [3] are same to the case of Hopf algebra.

**Definition 2.6** (see [31]) Let  $(C, \beta)$  be a Hom-coalgebra. A left  $(C, \beta)$ -Hom-comodule is a triple  $(M, \rho, \alpha)$ , where  $M$  is a linear space,  $\rho : M \rightarrow C \otimes M$  (write  $\rho(m) = m_{-1} \otimes m_0$ ,  $\forall m \in M$ ) is a linear map, and  $\alpha$  is an automorphism of  $M$ , such that

$$(HCM1) \quad \alpha(m)_{-1} \otimes \alpha(m)_0 = \beta(m_{-1}) \otimes \alpha(m_0),$$

$$(HCM2) \quad \beta(m_{-1}) \otimes m_{0-1} \otimes m_{00} = m_{-11} \otimes m_{-12} \otimes \alpha(m_0); \quad \varepsilon_C(m_{-1})m_0 = \alpha(m)$$

are satisfied for all  $m \in M$ .

Let  $(M, \rho^M, \alpha_M)$  and  $(N, \rho^N, \alpha_N)$  be two left  $(C, \beta)$ -Hom-comodules. Then a linear map  $f : M \rightarrow N$  is called a map of left  $(C, \beta)$ -Hom-comodules if  $f(m)_{-1} \otimes f(m)_0 = m_{-1} \otimes f(m_0)$  and  $\alpha_M \circ f = f \circ \alpha_N$ .

**Remarks** (1) It is obvious that  $(C, \Delta_C, \beta)$  is a left  $(C, \beta)$ -Hom-comodule.

(2) When  $\beta = id_A$  and  $\alpha = id_M$ , a left  $(C, \beta)$ -Hom-comodule is the usual left  $C$ -comodule.

**Definition 2.7** (see [31]) Let  $(H, \beta)$  be a Hom-bialgebra and  $(C, \alpha)$  a Hom-coalgebra. If  $(C, \rho, \alpha)$  is a left  $(H, \beta)$ -Hom-comodule and for all  $c \in C$ ,

$$(HCMC1) \quad \beta^2(c_{-1}) \otimes c_{01} \otimes c_{02} = c_{1-1}c_{2-1} \otimes c_{10} \otimes c_{20},$$

$$(HCMC2) \quad c_{-1}\varepsilon_C(c_0) = 1_H\varepsilon_C(c),$$

then  $(C, \rho, \alpha)$  is called an  $(H, \beta)$ -comodule Hom-coalgebra.

**Remarks** (1) When  $\alpha = id_A$  and  $\beta = id_H$ , an  $(H, \beta)$ -comodule Hom-coalgebra is the usual  $H$ -comodule coalgebra.

(2) Similar to the case of Hopf algebras, in [31], Zhang concluded that the Eq.(HCMC1) is satisfied if and only if  $\Delta_C$  is a morphism of  $H$ -comodules for suitable  $H$ -comodule structures on  $C \otimes C$  and  $C$ , respectively.

**Definition 2.8** (see [11]) Let  $(H, \beta)$  be a Hom-bialgebra and  $(C, \alpha)$  a Hom-coalgebra. If  $(C, \triangleright, \alpha)$  is a left  $(H, \beta)$ -Hom-module and for all  $h \in H$  and  $c \in A$ ,

$$(HMC1) \quad (h \triangleright c)_1 \otimes (h \triangleright c)_2 = (h_1 \triangleright c_1) \otimes (h_2 \triangleright c_2),$$

$$(HMC2) \quad \varepsilon_C(h \triangleright c) = \varepsilon_H(h)\varepsilon_C(c),$$

then  $(C, \triangleright, \alpha)$  is called an  $(H, \beta)$ -module Hom-coalgebra.

**Remark** When  $\alpha = id_C$  and  $\beta = id_H$ , an  $(H, \beta)$ -module Hom-coalgebra is the usual  $H$ -module coalgebra.

**Definition 2.9** (see [25]) Let  $(H, \beta)$  be a Hom-bialgebra and  $(A, \alpha)$  a Hom-algebra. If  $(A, \rho, \alpha)$  is a left  $(H, \beta)$ -Hom-comodule and for all  $a, a' \in A$ ,

$$(HCMA1) \quad \rho(aa') = a_{-1}a'_{-1} \otimes a_0a'_0,$$

$$(HCMA2) \quad \rho(1_A) = 1_H \otimes 1_A,$$

then  $(A, \rho, \alpha)$  is called an  $(H, \beta)$ -comodule Hom-algebra.

**Remark** When  $\alpha = id_A$  and  $\beta = id_H$ , an  $(H, \beta)$ -comodule Hom-algebra is the usual  $H$ -comodule algebra.

**Definition 2.10** (see [11]) Let  $(H, \beta)$  be a Hom-bialgebra and  $(A, \triangleright, \alpha)$  an  $(H, \beta)$ -module Hom-algebra. Then  $(A \sharp H, \alpha \otimes \beta)$  ( $A \sharp H = A \otimes H$  as a linear space) with the multiplication

$$(a \otimes h)(a' \otimes h') = a(h_1 \triangleright \alpha^{-1}(a')) \otimes \beta^{-1}(h_2)h',$$

where  $a, a' \in A, h, h' \in H$ , and unit  $1_A \otimes 1_H$  is a Hom-algebra, we call it smash product Hom-algebra denoted by  $(A \sharp H, \alpha \otimes \beta)$ .

**Remarks** Here the multiplication of smash product Hom-algebra is different from the one defined by Makhoul and Panaite in Theorem 3.1 in [17].

**Definition 2.11** (see [1, 15, 16]) Let  $H$  be a bialgebra and  $M$  a linear space which is a left  $H$ -module with action  $\triangleright : H \otimes M \longrightarrow M, h \otimes m \mapsto h \triangleright m$  and a left  $H$ -comodule with coaction  $\rho : M \longrightarrow H \otimes M, \rho(m) = m_{-1} \otimes m_0$ . Then  $M$  is called a (left-left) Yetter-Drinfeld module over  $H$  if the following compatibility condition holds, for all  $h \in H$  and  $m \in M$ ,

$$(YD) \quad h_1 m_{-1} \otimes (h_2 \triangleright m_0) = (h_1 \triangleright m)_{-1} h_2 \otimes (h_1 \triangleright m)_0.$$

When  $H$  is a Hopf algebra, then the condition  $(YD)$  is equivalent to

$$(YD)' \quad h_1 m_{-1} S_H(h_3) \otimes (h_2 \triangleright m_0) = (h \triangleright m)_{-1} \otimes (h \triangleright m)_0.$$

### 3 Radford biproduct Hom-Hopf algebra

In this section, we mainly generalize the Radford biproduct bialgebra in [20, Theorem 1] to the Hom-setting.

Dual to the Definition 2.10, we have:

**Proposition 3.1** Let  $(H, \beta)$  be a Hom-bialgebra and  $(C, \rho, \alpha)$  an  $(H, \beta)$ -comodule Hom-coalgebra. Then  $(C \diamond H, \alpha \otimes \beta)$  ( $C \diamond H = C \otimes H$  as a linear space) with the comultiplication

$$\Delta_{C \diamond H}(c \otimes h) = c_1 \otimes c_{2-1} \beta^{-1}(h_1) \otimes \alpha^{-1}(c_{20}) \otimes h_2,$$

where  $c \in C, h \in H$ , and counit  $\varepsilon_C \otimes \varepsilon_H$  is a Hom-coalgebra, we call it smash coproduct Hom-coalgebra denoted by  $(C \diamond H, \alpha \otimes \beta)$ .

In fact, dual to Theorem 3.1 in [11], we have

**Proposition 3.2** Let  $(C, \Delta_C, \varepsilon_C, \alpha)$  and  $(H, \Delta_H, \varepsilon_H, \beta)$  be two Hom-coalgebras,  $T : C \otimes H \longrightarrow H \otimes C$  (write  $T(c \otimes h) = h_T \otimes c_T, \forall c \in C, h \in H$ ) a linear map such that for all  $c \in C, h \in H$ ,

$$\alpha(c)_T \otimes \beta(h)_T = \alpha(c_T) \otimes \beta(h_T).$$

Then  $(C \diamond_T H, \alpha \otimes \beta)$  ( $C \diamond_T H = C \otimes H$  as a linear space) with the comultiplication

$$\Delta_{C \diamond_T H}(c \otimes h) = c_1 \otimes \beta^{-1}(h_1)_T \otimes \alpha^{-1}(c_{2T}) \otimes h_2,$$

and counit  $\varepsilon_C \otimes \varepsilon_H$  becomes a Hom-coalgebra if and only if the following conditions hold:

$$\begin{aligned} (C1) \quad & \varepsilon_H(h_T)c_T = \varepsilon_H(h)\alpha(c); \quad h_T\varepsilon_C(c_T) = \beta(h)\varepsilon_C(c), \\ (C2) \quad & h_{T1} \otimes h_{T2} \otimes \alpha(c_T) = \beta(\beta^{-1}(h_1)_T) \otimes h_{2t} \otimes c_{Tt}, \\ (C3) \quad & \beta(h_T) \otimes \alpha(c)_{T1} \otimes \alpha(c)_{T2} = h_{Tt} \otimes \alpha(c_1)_t \otimes \alpha(c_{2T}), \end{aligned}$$

where  $c \in C, h \in H$  and  $t$  is a copy of  $T$ .

We call this Hom-coalgebra  $T$ -smash coproduct Hom-coalgebra and denote it by  $(C \diamond_T H, \alpha \otimes \beta)$ .

**Remarks** (1) Let  $T(c \otimes h) = c_{-1}h \otimes c_0$  in  $C \diamond_T H$ , we can get the smash coproduct Hom-coalgebra  $C \diamond H$ .

(2) Here the comultiplication of  $T$ -smash coproduct Hom-coalgebra is slightly different from the one defined by Zheng in [32]. And the conditions (C1)–(C3) are simpler than the ones in [32].

**Theorem 3.3** Let  $(H, \beta)$  be a Hom-bialgebra,  $(A, \alpha)$  a left  $(H, \beta)$ -module Hom-algebra with module structure  $\triangleright : H \otimes A \rightarrow A$  and a left  $(H, \beta)$ -comodule Hom-coalgebra with comodule structure  $\rho : A \rightarrow H \otimes A$ . Then the following are equivalent:

- $(A_{\diamond H}^{\natural}, \mu_{A_{\diamond H}^{\natural}}, 1_A \otimes 1_H, \Delta_{A \diamond H}, \varepsilon_A \otimes \varepsilon_H, \alpha \otimes \beta)$  is a Hom-bialgebra, where  $(A_{\natural}^{\natural}H, \alpha \otimes \beta)$  is a smash product Hom-algebra and  $(A \diamond H, \alpha \otimes \beta)$  is a smash coproduct Hom-coalgebra.
- The following conditions hold ( $\forall a, b \in A$  and  $h \in H$ ):
  - (R1)  $(A, \rho, \alpha)$  is an  $(H, \beta)$ -comodule Hom-algebra,
  - (R2)  $(A, \triangleright, \alpha)$  is an  $(H, \beta)$ -module Hom-coalgebra,
  - (R3)  $\varepsilon_A$  is a Hom-algebra map and  $\Delta_A(1_A) = 1_A \otimes 1_A$ ,
  - (R4)  $\Delta_A(ab) = a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(b_1)) \otimes \alpha^{-1}(a_{20})b_2$ ,
  - (R5)  $h_1\beta(a_{-1}) \otimes (\beta^3(h_2) \triangleright a_0) = (\beta^2(h_1) \triangleright a)_{-1}h_2 \otimes (\beta^2(h_1) \triangleright a)_0$ .

In this case, we call this Hom-bialgebra Radford biproduct Hom-bialgebra and denote it by  $(A_{\diamond H}^{\natural}, \alpha \otimes \beta)$ .

**Proof** ( $\Leftarrow$ ) It is easy to prove that  $\varepsilon_{A_{\diamond H}^{\natural}} = \varepsilon_A \otimes \varepsilon_H$  is a morphism of Hom-algebras. Next we check  $\Delta_{A_{\diamond H}^{\natural}} = \Delta_{A \diamond H}$  is a morphism of Hom-algebras as follows. For all  $a, b \in A$  and  $h, g \in H$ , we have

$$\begin{aligned} & \Delta_{A_{\diamond H}^{\natural}}((a \otimes h)(b \otimes g)) \\ &= (a(h_1 \triangleright \alpha^{-1}(b)))_1 \otimes (a(h_1 \triangleright \alpha^{-1}(b)))_{2-1} \beta^{-1}((\beta^{-1}(h_2)g)_1) \\ & \quad \otimes \alpha^{-1}((a(h_1 \triangleright \alpha^{-1}(b)))_{20}) \otimes (\beta^{-1}(h_2)g)_2 \end{aligned}$$

$$\begin{aligned}
& \stackrel{(HA1)(HC1)}{=} (a(h_1 \triangleright \alpha^{-1}(b)))_1 \otimes (a(h_1 \triangleright \alpha^{-1}(b)))_{2-1} (\beta^{-2}(h_{21})\beta^{-1}(g_1)) \\
& \quad \otimes \alpha^{-1}((a(h_1 \triangleright \alpha^{-1}(b)))_{20}) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(R4)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}((h_1 \triangleright \alpha^{-1}(b))_1)) \otimes (\alpha^{-1}(a_{20})(h_1 \triangleright \alpha^{-1}(b))_{2-1}) \\
& \quad \times (\beta^{-2}(h_{21})\beta^{-1}(g_1)) \otimes \alpha^{-1}((\alpha^{-1}(a_{20})(h_1 \triangleright \alpha^{-1}(b))_2)_0) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HCA1)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}((h_1 \triangleright \alpha^{-1}(b))_1)) \otimes (\alpha^{-1}(a_{20})_{-1}(h_1 \triangleright \alpha^{-1}(b))_{2-1}) \\
& \quad \times (\beta^{-2}(h_{21})\beta^{-1}(g_1)) \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((h_1 \triangleright \alpha^{-1}(b))_{20}) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HMC1)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(h_{11} \triangleright \alpha^{-1}(b_1))) \otimes (\alpha^{-1}(a_{20})_{-1}(h_{12} \triangleright \alpha^{-1}(b_2))_{-1}) \\
& \quad \times (\beta^{-2}(h_{21})\beta^{-1}(g_1)) \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((h_{12} \triangleright \alpha^{-1}(b_2))_0) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HA2)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(h_{11} \triangleright \alpha^{-1}(b_1))) \otimes (\alpha^{-1}(a_{20})_{-1}\beta^{-1}((h_{12} \triangleright \alpha^{-1}(b_2))_{-1}) \\
& \quad \times (\beta^{-2}(h_{21}))g_1 \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((h_{12} \triangleright \alpha^{-1}(b_2))_0) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HC2)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1))) \otimes (\alpha^{-1}(a_{20})_{-1}) \\
& \quad \times \beta^{-1}((\beta^{-1}(h_{211}) \triangleright \alpha^{-1}(b_2))_{-1}\beta^{-3}(h_{212}))g_1 \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0) \\
& \quad \times \alpha^{-1}((\beta^{-1}(h_{211}) \triangleright \alpha^{-1}(b_2))_0) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HC1)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1))) \otimes (\alpha^{-1}(a_{20})_{-1}) \\
& \quad \times \beta^{-1}((\beta^2(\beta^{-3}(h_{21})_1) \triangleright \alpha^{-1}(b_2))_{-1}\beta^{-3}(h_{212}))g_1 \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0) \\
& \quad \times \alpha^{-1}((\beta^2(\beta^{-3}(h_{21})_1) \triangleright \alpha^{-1}(b_2))_0) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(R5)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1))) \otimes (\alpha^{-1}(a_{20})_{-1}\beta^{-1}(\beta^{-3}(h_{21})_1) \\
& \quad \times \beta(\alpha^{-1}(b_2)_{-1}))g_1 \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}(\beta^3(\beta^{-3}(h_{21})_2) \triangleright \alpha^{-1}(b_2)_0) \\
& \stackrel{(HCM1)(HC1)}{=} a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1))) \otimes (\beta^{-1}(a_{20-1})\beta^{-1}(\beta^{-3}(h_{211}) \\
& \quad \times b_{2-1}))g_1 \otimes \alpha^{-2}(a_{200})\alpha^{-1}(h_{212} \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HCM2)}{=} a_1(\beta(a_{2-11}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1))) \otimes (\beta^{-1}(a_{2-12})\beta^{-1}(\beta^{-3}(h_{211}) \\
& \quad \times b_{2-1}))g_1 \otimes \alpha^{-1}(a_{20})\alpha^{-1}(h_{212} \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HA2)}{=} a_1(\beta(a_{2-11}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1))) \otimes (\beta^{-1}(a_{2-12})\beta^{-3}(h_{211})) \\
& \quad \times (b_{2-1}\beta^{-1}(g_1)) \otimes \alpha^{-1}(a_{20})\alpha^{-1}(h_{212} \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HC2)}{=} a_1(\beta(a_{2-11}) \triangleright \alpha^{-1}(h_{11} \triangleright \alpha^{-1}(b_1))) \otimes (\beta^{-1}(a_{2-12})\beta^{-2}(h_{12}))(b_{2-1}\beta^{-1}(g_1)) \\
& \quad \otimes \alpha^{-1}(a_{20})\alpha^{-1}(\beta(h_{21}) \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HM1)}{=} a_1(\beta(a_{2-11}) \triangleright (\beta^{-1}(h_{11}) \triangleright \alpha^{-2}(b_1))) \otimes (\beta^{-1}(a_{2-12})\beta^{-2}(h_{12}))(b_{2-1}\beta^{-1}(g_1)) \\
& \quad \otimes \alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HM2)}{=} a_1((a_{2-11}\beta^{-1}(h_{11})) \triangleright \alpha^{-1}(b_1)) \otimes (\beta^{-1}(a_{2-12})\beta^{-2}(h_{12}))(b_{2-1}\beta^{-1}(g_1)) \\
& \quad \otimes \alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\
& \stackrel{(HA1)}{=} a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright \alpha^{-1}(b_1)) \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2)(b_{2-1}\beta^{-1}(g_1))
\end{aligned}$$



$$\begin{aligned}
& \otimes \alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\
&= (a_1 \otimes a_{2-1}\beta^{-1}(h_1) \otimes \alpha^{-1}(a_{20}) \otimes h_2)(b_1 \otimes b_{2-1}\beta^{-1}(h_1) \otimes \alpha^{-1}(b_{20}) \otimes h_2) \\
&= \Delta_{A \bowtie H}(a \otimes h) \Delta_{A \bowtie H}(b \otimes g),
\end{aligned}$$

and  $\Delta_{A \bowtie H}(1_A \otimes 1_H) = 1_A \otimes 1_H \otimes 1_A \otimes 1_H$  can be proved directly.

( $\implies$ ) We only verify that the conditions (R4) and (R5) hold, and others hold similarly. Since  $\Delta_{A \bowtie H} = \Delta_{A \diamond H}$  is a morphism of Hom-algebras, for all  $a, b \in A$  and  $h, g \in H$ , we have

$$\begin{aligned}
& a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright \alpha^{-1}(b_1)) \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2)(b_{2-1}\beta^{-1}(g_1)) \\
& \otimes \alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 = (a(h_1 \triangleright \alpha^{-1}(b)))_1 \otimes (a(h_1 \triangleright \alpha^{-1}(b)))_{2-1} \\
& \times \beta^{-1}((\beta^{-1}(h_2)g)_1) \otimes \alpha^{-1}((a(h_1 \triangleright \alpha^{-1}(b)))_{20}) \otimes (\beta^{-1}(h_2)g)_2
\end{aligned}$$

Then, apply  $id_A \otimes \varepsilon_H \otimes id_A \otimes \varepsilon_H$  to the above equation and set  $h = g = 1_H$ , we get (HB). (HYD) can be obtained by using  $\varepsilon_A \otimes id_H \otimes id_A \otimes \varepsilon_H$  to the above equation and setting  $a = 1_A, g = 1_H$ .  $\square$

**Remarks** (1) If  $\alpha = id_A$  and  $\beta = id_H$ , then we can get the well-known Radford biproduct bialgebra in [20, Theorem 1].

(2) Theorem 3.3 is different from the one defined by Liu and Shen in [9], because the Hom-smash product there is based on the concept of module Hom-algebra in [3] and ours is based on the Yau's in [24, 29].

**Corollary 3.4** (see [11]) Let  $(A, \alpha), (H, \beta)$  be two Hom-bialgebras, and  $(A, \triangleright, \alpha)$  an  $(H, \beta)$ -module Hom-algebra. Then the smash product Hom-algebra  $(A \sharp H, \alpha \otimes \beta)$  endowed with the tensor product Hom-coalgebra structure becomes a Hom-bialgebra if and only if  $(A, \triangleright, \alpha)$  is an  $(H, \beta)$ -module Hom-coalgebra and

$$h_1 \otimes h_2 \triangleright a = h_2 \otimes h_1 \triangleright a.$$

**Proof** Let the comodule action  $\rho$  be trivial, i.e.  $\rho(a) = 1_H \otimes \alpha(a)$  in Theorem 3.3.  $\square$

**Corollary 3.5** Let  $(C, \alpha), (H, \beta)$  be two Hom-bialgebras, and  $(C, \rho, \alpha)$  an  $(H, \beta)$ -comodule Hom-coalgebra. Then the smash coproduct Hom-coalgebra  $(C \diamond H, \alpha \otimes \beta)$  endowed with the tensor product Hom-algebra structure becomes a Hom-bialgebra if and only if  $(C, \rho, \alpha)$  is an  $(H, \beta)$ -comodule Hom-algebra and

$$hc_{-1} \otimes c_0 = c_{-1}h \otimes c_0.$$

**Proof** Let the module action  $\triangleright$  be trivial, i.e.  $h \triangleright c = \varepsilon_H(h)\alpha(c)$  in Theorem 3.3.  $\square$

**Theorem 3.6** Let  $(H, \beta, S_H)$  be a Hom-Hopf algebra, and  $(A, \alpha)$  be a Hom-algebra and a Hom-coalgebra. Assume that  $(A \sharp H, \alpha \otimes \beta)$  is a Radford biproduct Hom-bialgebra defined

as above, and  $S_A : A \longrightarrow A$  is a linear map such that  $S_A(a_1)a_2 = a_1S_A(a_2) = \varepsilon_A(a)1_A$  and  $\alpha \circ S_A = S_A \circ \alpha$  hold. Then  $(A_{\diamond}^{\natural}H, \alpha \otimes \beta, S_{A_{\diamond}^{\natural}H})$  is a Hom-Hopf algebra, where

$$S_{A_{\diamond}^{\natural}H}(a \otimes h) = (S_H(a_{-1}\beta^{-1}(h))_1 \triangleright S_A(\alpha^{-2}(a_0))) \otimes \beta^{-1}(S_H(a_{-1}\beta^{-1}(h))_2).$$

**Proof** We can compute that  $(A_{\diamond}^{\natural}H, \alpha \otimes \beta, S_{A_{\diamond}^{\natural}H})$  is a Hom-Hopf algebra as follows. For all  $a \in A$  and  $h \in H$ , we have

$$\begin{aligned} & (S_{A_{\diamond}^{\natural}H} * id_{A_{\diamond}^{\natural}H})(a \otimes h) \\ &= (S_H(a_{1-1}\beta^{-1}(a_{2-1}\beta^{-1}(h_1)))_1 \triangleright S_A(\alpha^{-2}(a_{10}))) (\beta^{-1}(S_H(a_{1-1}\beta^{-1}(a_{2-1}\beta^{-1}(h_1)))_2)_1 \triangleright \alpha^{-2}(a_{20})) \otimes \beta^{-1}(\beta^{-1}(S_H(a_{1-1}\beta^{-1}(a_{2-1}\beta^{-1}(h_1)))_2)_2) h_2 \\ & \stackrel{(HA1)(HA2)}{=} (S_H(\beta^{-1}(a_{1-1}a_{2-1})\beta^{-1}(h_1))_1 \triangleright S_A(\alpha^{-2}(a_{10}))) (\beta^{-1}(S_H(\beta^{-1}(a_{1-1}a_{2-1})\beta^{-1}(h_1))_2)_1 \triangleright \alpha^{-2}(a_{20})) \otimes \beta^{-1}(\beta^{-1}(S_H(\beta^{-1}(a_{1-1}a_{2-1})\beta^{-1}(h_1))_2)_2) h_2 \\ & \stackrel{(HCMC1)}{=} (S_H(\beta(a_{-1})\beta^{-1}(h_1))_1 \triangleright S_A(\alpha^{-2}(a_{01}))) (\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2)_1 \triangleright \alpha^{-2}(a_{02})) \otimes \beta^{-1}(\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2)_2) h_2 \\ & \stackrel{(HC1)(HC2)}{=} (\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_{11}) \triangleright S_A(\alpha^{-2}(a_{01}))) (\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_{12}) \triangleright \alpha^{-2}(a_{02})) \otimes \beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2) h_2 \\ & \stackrel{(HC1)(HMA1)}{=} (\beta(S_H(\beta(a_{-1})\beta^{-1}(h_1))_1) \triangleright (S_A(\alpha^{-2}(a_{01}))\alpha^{-2}(a_{02}))) \otimes \beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2) h_2 \\ & \stackrel{(HA1)}{=} (\beta(S_H(\beta(a_{-1})\beta^{-1}(h_1))_1) \triangleright 1_A \varepsilon_A(a_0)) \otimes \beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2) h_2 \\ & \stackrel{(HCMC2)}{=} (\beta(S_H(h_1)_1) \triangleright 1_A \varepsilon_A(a)) \otimes \beta^{-1}(S_H(h_1)_2) h_2 \\ & \stackrel{(HMA2)}{=} 1_A \varepsilon_A(a) \otimes S_H(h_1) h_2 \\ &= (1_A \otimes 1_H) \varepsilon_A(a) \varepsilon_H(h) \end{aligned}$$

and

$$\begin{aligned} & (id_{A_{\diamond}^{\natural}H} * S_{A_{\diamond}^{\natural}H})(a \otimes h) \\ &= a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright \alpha^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_1 \triangleright S_A(\alpha^{-2}(\alpha^{-1}(a_{20})_0)))) \\ & \quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2) \beta^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_2) \\ & \stackrel{(HM1)}{=} a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright (\beta^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_1) \triangleright S_A(\alpha^{-3}(\alpha^{-1}(a_{20})_0)))) \\ & \quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2) \beta^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_2) \\ & \stackrel{(HM2)(HA1)}{=} a_1(\beta^{-1}((a_{2-1}\beta^{-1}(h_1))_1) S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_1 \triangleright S_A(\alpha^{-2}(\alpha^{-1}(a_{20})_0))) \\ & \quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2) S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_2 \\ & \stackrel{(HC1)}{=} a_1(\beta^{-1}((a_{2-1}\beta^{-1}(h_1)) S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2)))_1 \triangleright S_A(\alpha^{-2}(\alpha^{-1}(a_{20})_0))) \\ & \quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1)) S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2)))_2 \end{aligned}$$

$$\begin{aligned}
& \stackrel{(HCM1)}{=} a_1(\beta^{-1}((a_{2-1}\beta^{-1}(h_1))S_H(\beta^{-1}(a_{20-1})\beta^{-1}(h_2)))_1 \triangleright S_A(\alpha^{-3}(a_{200}))) \\
& \quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))S_H(\beta^{-1}(a_{20-1})\beta^{-1}(h_2)))_2 \\
& \stackrel{(HCM2)}{=} a_1(\beta^{-1}((\beta^{-1}(a_{2-11})\beta^{-1}(h_1))S_H(\beta^{-1}(a_{2-12})\beta^{-1}(h_2)))_1 \triangleright S_A(\alpha^{-2}(a_{20}))) \\
& \quad \otimes \beta^{-1}((\beta^{-1}(a_{2-11})\beta^{-1}(h_1))S_H(\beta^{-1}(a_{2-12})\beta^{-1}(h_2)))_2 \\
& \stackrel{(HC1)}{=} a_1((1_H \triangleright S_A(\alpha^{-2}(a_{20})))\varepsilon_H(a_{2-1}) \otimes 1_H \varepsilon_H(h)) \\
& \stackrel{(HCM2)}{=} a_1(1_H \triangleright S_A(\alpha^{-1}(a_2))) \otimes 1_H \varepsilon_H(h) \\
& \stackrel{(HM2)}{=} a_1 S_A(a_2) \otimes 1_H \varepsilon_H(h) \\
& = (1_A \otimes 1_H) \varepsilon_A(a) \varepsilon_H(h),
\end{aligned}$$

while

$$\begin{aligned}
& S_{A \bowtie H}(\alpha(a) \otimes \beta(h)) \\
& = (S_H(\alpha(a)_{-1}h)_1 \triangleright S_A(\alpha^{-2}(\alpha(a)_0))) \otimes \beta^{-1}(S_H(\alpha(a)_{-1}h)_2) \\
& \stackrel{(HCM1)}{=} (S_H(\beta(a_{-1})h)_1 \triangleright S_A(\alpha^{-1}(a_0))) \otimes \beta^{-1}(S_H(\beta(a_{-1})h)_2) \\
& = (\alpha \otimes \beta)(S_{A \bowtie H}(a \otimes h)),
\end{aligned}$$

finishing the proof.  $\square$

**Corollary 3.7**(see [11]) Let  $(A, \alpha, S_A), (H, \beta, S_H)$  be two Hom-Hopf algebras, and  $(A \bowtie H, \alpha \otimes \beta)$  a smash product Hom-bialgebra. Then  $(A \bowtie H, \alpha \otimes \beta, S_{A \bowtie H})$  is a Hom-Hopf algebra, where

$$S_{A \bowtie H}(a \otimes h) = (S_H(h)_1 \triangleright \alpha^{-1}(S_A(a))) \otimes \beta^{-1}(S_H(h)_2).$$

**Proof** Let the comodule action  $\rho$  be trivial, i.e.  $\rho(a) = 1_H \otimes \alpha(a)$  in Theorem 3.6.  $\square$

**Corollary 3.8** Let  $(C, \alpha, S_C), (H, \beta, S_H)$  be two Hom-Hopf algebras, and  $(C \diamond H, \alpha \otimes \beta)$  a smash coproduct Hom-bialgebra. Then  $(C \diamond H, \alpha \otimes \beta, S_{C \diamond H})$  is a Hom-Hopf algebra, where

$$S_{C \diamond H}(c \otimes h) = S_C(\alpha^{-1}(c_{(0)})) \otimes S_H(c_{(-1)}\beta^{-1}(h)).$$

**Proof** Let the module action  $\triangleright$  be trivial, i.e.  $h \triangleright c = \varepsilon_H(h)\alpha(c)$  in Theorem 3.6.  $\square$

## 4 Hom-Yetter-Drinfeld category

In this section, we give the definition of Hom-Yetter-Drinfeld module and also prove that the category  ${}^H_H\mathbb{YD}$  of Hom-Yetter-Drinfeld modules is a pre-braided tensor category. Furthermore, we obtain that  $(A \bowtie H, \alpha \otimes \beta)$  is a Radford biproduct Hom-bialgebra if and only if  $(A, \alpha)$  is a Hom-bialgebra in the category  ${}^H_H\mathbb{YD}$ .

**Definition 4.1** Let  $(H, \beta)$  be a Hom-bialgebra,  $(M, \triangleright_M, \alpha_M)$  a left  $(H, \beta)$ -module with action  $\triangleright_M : H \otimes M \longrightarrow M, h \otimes m \mapsto h \triangleright_M m$  and  $(M, \rho^M, \alpha_M)$  a left  $(H, \beta)$ -comodule with coaction  $\rho^M : M \longrightarrow H \otimes M, m \mapsto m_{-1} \otimes m_0$ . Then we call  $(M, \triangleright_M, \rho^M, \alpha_M)$  a (left-left) Hom-Yetter-Drinfeld module over  $(H, \beta)$  if the following condition holds:

$$(HYD) \quad h_1 \beta(m_{-1}) \otimes (\beta^3(h_2) \triangleright_M m_0) = (\beta^2(h_1) \triangleright_M m)_{-1} h_2 \otimes (\beta^2(h_1) \triangleright_M m)_0,$$

where  $h \in H$  and  $m \in M$ .

**Remarks**(1) The compatibility condition  $(HYD)$  is different from the condition (2.1) in [16, Definition 2.1], the condition (3.1) in [5, Definition 3.1] and the condition (4.1) in [9, Definition 4.1].

(2) When  $\beta = id_H$ , the condition  $(HYD)$  is exactly the condition  $(YD)$ .

(3) Let  $(H, \beta)$  be a Hom-bialgebra and  $K$  a field. Then  $(K, id_K)$  is a (left-left) Hom-Yetter-Drinfeld module over  $(H, \beta)$  with the module and comodule actions defined as follows:  $H \otimes K \longrightarrow K, h \otimes k \mapsto \varepsilon(h)k$  and  $K \longrightarrow H \otimes K, k \mapsto 1_H \otimes k$ .

(4) When  $(H, \beta, S_H)$  is a Hom-Hopf algebra, then the condition  $(HYD)$  is equivalent to

$$(HYD)' \quad (\beta^4(h) \triangleright_M m)_{-1} \otimes (\beta^4(h) \triangleright_M m)_0 = \beta^{-2}(h_{11} \beta(m_{-1})) S_H(h_2) \otimes (\beta^3(h_{12}) \triangleright_M m_0).$$

**Proof**  $(\implies) \quad \beta^{-2}(h_{11} \beta(m_{-1})) S(h_2) \otimes (\beta^3(h_{12}) \triangleright m_0)$

$$\stackrel{(HYD)}{=} \beta^{-2}((\beta^2(h_{11} \triangleright m))_{-1} h_{12}) S(h_2) \otimes (\beta^2(h_{11} \triangleright m))_0$$

$$\stackrel{(HA1)(HA2)}{=} \beta^{-1}((\beta^2(h_{11} \triangleright m))_{-1}) (\beta^{-2}(h_{12}) \beta^{-1}(S(h_2))) \otimes (\beta^2(h_{11} \triangleright m))_0$$

$$\stackrel{(HC2)}{=} \beta^{-1}((\beta^2(h_1 \triangleright m))_{-1}) (\beta^{-2}(h_{21}) \beta^{-2}(S(h_{22}))) \otimes (\beta^2(h_1 \triangleright m))_0$$

$$\stackrel{(HA1)}{=} \beta^{-1}((\beta^2(h_1 \triangleright m))_{-1}) (\beta^{-2}(h_{21} S(h_{22}))) \otimes (\beta^2(h_1 \triangleright m))_0$$

$$\stackrel{(HA2)(HC2)}{=} (\beta^4(h) \triangleright m)_{-1} \otimes (\beta^4(h) \triangleright m)_0.$$

$$(\impliedby) \quad (\beta^2(h_1) \triangleright m)_{-1} h_2 \otimes (\beta^2(h_1) \triangleright m)_0$$

$$\stackrel{(HYD)'}{=} (\beta^{-2}(\beta^{-2}(h_1)_{11} \beta(m_{-1})) S(\beta^{-2}(h_1)_2)) h_2 \otimes (\beta^3(\beta^{-2}(h_1)_{12}) \triangleright m_0)$$

$$\stackrel{(HC1)}{=} (\beta^{-2}(\beta^{-2}(h_{111}) \beta(m_{-1})) S(\beta^{-2}(h_{12}))) h_2 \otimes (\beta(h_{112}) \triangleright m_0)$$

$$\stackrel{(HC2)(HC1)}{=} (\beta^{-2}(\beta^{-1}(h_{11}) \beta(m_{-1})) S(\beta^{-2}(h_{21}))) \beta^{-1}(h_{22}) \otimes (\beta^2(h_{12}) \triangleright m_0)$$

$$\stackrel{(HA2)(HA1)}{=} (\beta^{-1}(\beta^{-1}(h_{11}) \beta(m_{-1})) (\beta^{-2} S(h_{21}) h_{22})) \otimes (\beta^2(h_{12}) \triangleright m_0)$$

$$= (\beta^{-1}(\beta^{-1}(h_{11}) \beta(m_{-1})) 1_H \varepsilon_H(h_2)) \otimes (\beta^2(h_{12}) \triangleright m_0)$$

$$\stackrel{(HC1)(HC2)(HA1)}{=} h_1 \beta(m_{-1}) \otimes (\beta^3(h_2) \triangleright m_0).$$

Here we use  $\triangleright, S$  instead of  $\triangleright_M, S_H$ , respectively.  $\square$

**Definition 4.2** Let  $(H, \beta)$  be a Hom-bialgebra. We denote by  ${}^H_H \mathbb{YD}$  the category whose objects are Hom-Yetter-Drinfeld modules  $(M, \triangleright_M, \rho^M, \alpha_M)$  over  $(H, \beta)$ ; the morphisms in the category are morphisms of left  $(H, \beta)$ -modules and left  $(H, \beta)$ -comodules.

In the following, we give a solution of Hom-Yang-Baxter equation introduced and studied by Yau in [26, 27, 30].

**Proposition 4.3** Let  $(H, \beta)$  be a Hom-bialgebra and  $(M, \triangleright_M, \rho^M, \alpha_M), (N, \triangleright_N, \rho^N, \alpha_N) \in {}^H_H \mathbb{YD}$ . Define the linear map

$$\tau_{M,N} : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto \beta^3(m_{-1}) \triangleright_N n \otimes m_0,$$

where  $m \in M$  and  $n \in N$ . Then, we have  $\tau_{M,N} \circ (\alpha_M \otimes \alpha_N) = (\alpha_N \otimes \alpha_M) \circ \tau_{M,N}$  and, if  $(P, \triangleright_P, \rho^P, \alpha_P) \in {}^H_H \mathbb{YD}$ , the maps  $\tau_{\_, \_}$  satisfy the Hom-Yang-Baxter equation:

$$(\alpha_P \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes \tau_{N,P}) = (\tau_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes \alpha_P).$$

**Proof** We only check that the second equality holds, and the first one is easy. For all  $m \in M, n \in N$  and  $p \in P$ , we have

$$\begin{aligned} & (\alpha_P \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes \tau_{N,P})(m \otimes n \otimes p) \\ &= (\beta^3(\alpha_M(m)_{-1}) \triangleright_P (\beta^3(n_{-1}) \triangleright_P p)) \otimes \beta^3(\alpha_M(m)_{0-1}) \triangleright_N \alpha_N(n_0) \otimes \alpha_M(m)_{00} \\ &\stackrel{(HM1)}{=} (\beta^4(\alpha_M(m)_{-1}) \triangleright_P (\beta^4(n_{-1}) \triangleright_P \alpha_P(p))) \otimes \beta^3(\alpha_M(m)_{0-1}) \triangleright_N \alpha_N(n_0) \\ &\quad \otimes \alpha_M(m)_{00} \\ &\stackrel{(HCM1)}{=} (\beta^5(m_{-1}) \triangleright_P (\beta^4(n_{-1}) \triangleright_P \alpha_P(p))) \otimes \beta^4(m_{0-1}) \triangleright_N \alpha_N(n_0) \otimes \alpha_M(m_{00}) \\ &\stackrel{(HCM2)}{=} (\beta^4(m_{-11}) \triangleright_P (\beta^4(n_{-1}) \triangleright_P \alpha_P(p))) \otimes \beta^4(m_{-12}) \triangleright_N \alpha_N(n_0) \otimes \alpha_M^2(m_0) \\ &\stackrel{(HM2)}{=} ((\beta^3(m_{-11}) \beta^4(n_{-1})) \triangleright_P \alpha_P^2(p)) \otimes \beta^4(m_{-12}) \triangleright_N \alpha_N(n_0) \otimes \alpha_M^2(m_0) \\ &\stackrel{(HCM1)}{=} ((\beta^3(m_{-11}) \alpha_N(n)_{-1})) \triangleright_P \alpha_P^2(p)) \otimes \beta^4(m_{-12}) \triangleright_N \alpha_N(n)_0 \otimes \alpha_M^2(m_0) \\ &\stackrel{(HA1)}{=} (\beta^2(\beta(m_{-11}) \beta(\alpha_N(n)_{-1}))) \triangleright_P \alpha_P^2(p)) \otimes \beta^3(\beta(m_{-12})) \triangleright_N \alpha_N(n)_0 \otimes \alpha_M^2(m_0) \\ &\stackrel{(HC1)}{=} (\beta^2(\beta(m_{-1})_1 \beta(\alpha_N(n)_{-1}))) \triangleright_P \alpha_P^2(p)) \otimes \beta^3(\beta(m_{-1})_2) \triangleright_N \alpha_N(n)_0 \otimes \alpha_M^2(m_0) \\ &\stackrel{(HYD)}{=} (\beta^2((\beta^2(\beta(m_{-1})_1) \triangleright_N \alpha_N(n))_{-1} \beta(m_{-1})_2) \triangleright_P \alpha_P^2(p)) \\ &\quad \otimes \beta^2(\beta(m_{-1})_1) \triangleright_N \alpha_N(n)_0 \otimes \alpha_M^2(m_0) \\ &\stackrel{(HA1)(HC1)}{=} ((\beta^2((\beta^3(m_{-11}) \triangleright_N \alpha_N(n))_{-1}) \beta^3(m_{-12})) \triangleright_P \alpha_P^2(p)) \\ &\quad \otimes (\beta^3(m_{-11}) \triangleright_N \alpha_N(n))_0 \otimes \alpha_M^2(m_0) \\ &\stackrel{(HCM2)}{=} ((\beta^2((\beta^4(m_{-1}) \triangleright_N \alpha_N(n))_{-1}) \beta^3(m_{0-1})) \triangleright_P \alpha_P^2(p)) \\ &\quad \otimes (\beta^4(m_{-1}) \triangleright_N \alpha_N(n))_0 \otimes \alpha_M(m_{00}) \\ &\stackrel{(HM2)}{=} (\beta^3((\beta^4(m_{-1}) \triangleright_N \alpha_N(n))_{-1}) \triangleright_P (\beta^3(m_{0-1}) \triangleright_P \alpha_P(p))) \\ &\quad \otimes (\beta^4(m_{-1}) \triangleright_N \alpha_N(n))_0 \otimes \alpha_M(m_{00}) \\ &\stackrel{(HM1)}{=} (\beta^3(\alpha_N(\beta^3(m_{-1}) \triangleright_N n)_{-1}) \triangleright_P (\beta^3(m_{0-1}) \triangleright_P \alpha_P(p))) \\ &\quad \otimes \alpha_N(\beta^3(m_{-1}) \triangleright_N n)_0 \otimes \alpha_M(m_{00}) \\ &= (\tau_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes \alpha_P)(m \otimes n \otimes p). \end{aligned}$$

□

**Lemma 4.4** Let  $(H, \beta)$  be a Hom-bialgebra and  $(M, \triangleright_M, \rho^M, \alpha_M)$ ,  $(N, \triangleright_N, \rho^N, \alpha_N) \in {}^H_H \mathbb{YD}$ . Define the linear maps

$$\triangleright_{M \otimes N} : H \otimes M \otimes N \longrightarrow M \otimes N, h \otimes m \otimes n \mapsto (h_1 \triangleright_M m) \otimes (h_2 \triangleright_N n),$$

and

$$\rho^{M \otimes N} : M \otimes N \longrightarrow H \otimes M \otimes N, m \otimes n \mapsto \beta^{-2}(m_{-1}n_{-1}) \otimes m_0 \otimes n_0,$$

where  $h \in H$ ,  $m \in M$  and  $n \in N$ . Then  $(M \otimes N, \triangleright_{M \otimes N}, \rho^{M \otimes N}, \alpha_M \otimes \alpha_N)$  is a Hom-Yetter-Drinfeld module.

**Proof** It is easy to check that  $(M \otimes N, \triangleright_{M \otimes N}, \alpha_M \otimes \alpha_N)$  is an  $(H, \beta)$ -Hom-module and  $(M \otimes N, \rho^{M \otimes N}, \alpha_M \otimes \alpha_N)$  is an  $(H, \beta)$ -Hom-comodule. While for  $h \in H$ ,  $m \in M$  and  $n \in N$ , we have

$$\begin{aligned} & (\beta^2(h_1) \triangleright_{M \otimes N} (m \otimes n))_{-1} h_2 \otimes (\beta^2(h_1) \triangleright_{M \otimes N} (m \otimes n))_0 \\ &= ((\beta^2(h_1)_1 \triangleright_M m) \otimes (\beta^2(h_1)_2 \triangleright_N n))_{-1} h_2 \\ & \quad \otimes ((\beta^2(h_1)_1 \triangleright_M m) \otimes (\beta^2(h_1)_2 \triangleright_N n))_0 \\ &= \beta^{-2}(((\beta^2(h_1)_1 \triangleright_M m)_{-1} (\beta^2(h_1)_2 \triangleright_N n)_{-1}) \beta^2(h_2)) \otimes (\beta^2(h_1)_1 \triangleright_M m)_0 \\ & \quad \otimes (\beta^2(h_1)_2 \triangleright_N n)_0 \\ & \stackrel{(HA1)(HA2)}{=} \beta^{-2}(\beta((\beta^2(h_{11}) \triangleright_M m)_{-1})((\beta^2(h_{12}) \triangleright_N n)_{-1} \beta(h_2))) \otimes (\beta^2(h_{11}) \triangleright_M m)_0 \\ & \quad \otimes (\beta^2(h_{12}) \triangleright_N n)_0 \\ & \stackrel{(HC2)}{=} \beta^{-2}(\beta((\beta^3(h_1) \triangleright_M m)_{-1})((\beta^2(h_{21}) \triangleright_N n)_{-1} h_{22})) \otimes (\beta^3(h_1) \triangleright_M m)_0 \\ & \quad \otimes (\beta^2(h_{21}) \triangleright_N n)_0 \\ & \stackrel{(HYD)}{=} \beta^{-2}(\beta((\beta^3(h_1) \triangleright_M m)_{-1})(h_{21} \beta(n_{-1}))) \otimes (\beta^3(h_1) \triangleright_M m)_0 \\ & \quad \otimes (\beta^3(h_{22}) \triangleright_N n)_0 \\ & \stackrel{(HA2)}{=} \beta^{-2}(((\beta^3(h_1) \triangleright_M m)_{-1} h_{21}) \beta^2(n_{-1})) \otimes (\beta^3(h_1) \triangleright_M m)_0 \otimes (\beta^3(h_{22}) \triangleright_N n)_0 \\ & \stackrel{(HC2)}{=} \beta^{-2}(((\beta^2(h_{11}) \triangleright_M m)_{-1} h_{12}) \beta^2(n_{-1})) \otimes (\beta^2(h_{11}) \triangleright_M m)_0 \\ & \quad \otimes (\beta^4(h_2) \triangleright_N n)_0 \\ & \stackrel{(HYD)}{=} \beta^{-2}((h_{11} \beta(m_{-1})) \beta^2(n_{-1})) \otimes (\beta^3(h_{12}) \triangleright_M m)_0 \otimes (\beta^4(h_2) \triangleright_N n)_0 \\ & \stackrel{(HA1)}{=} (\beta^{-2}(h_{11}) \beta^{-1}(m_{-1})) n_{-1} \otimes (\beta^3(h_{12}) \triangleright_M m)_0 \otimes (\beta^4(h_2) \triangleright_N n)_0 \\ & \stackrel{(HA2)}{=} \beta^{-1}(h_{11}) (\beta^{-1}(m_{-1}) \beta^{-1}(n_{-1})) \otimes (\beta^3(h_{12}) \triangleright_M m)_0 \otimes (\beta^4(h_2) \triangleright_N n)_0 \\ & \stackrel{(HC2)}{=} h_1 (\beta^{-1}(m_{-1}) \beta^{-1}(n_{-1})) \otimes (\beta^3(h_{21}) \triangleright_M m)_0 \otimes (\beta^3(h_{22}) \triangleright_N n)_0 \\ & \stackrel{(HC1)(HA1)}{=} h_1 \beta(\beta^{-2}(m_{-1} n_{-1})) \otimes (\beta^3(h_2)_1 \triangleright_M m)_0 \otimes (\beta^3(h_2)_2 \triangleright_N n)_0 \\ &= h_1 \beta((m \otimes n)_{-1}) \otimes (\beta^3(h_2) \triangleright_{M \otimes N} (m \otimes n))_0, \end{aligned}$$

thus, the condition (HYD) holds. Therefore  $(M \otimes N, \triangleright_{M \otimes N}, \rho^{M \otimes N}, \alpha_M \otimes \alpha_N)$  is a Hom-Yetter-Drinfeld module.  $\square$

**Lemma 4.5** Let  $(H, \beta)$  be a Hom-bialgebra and  $(M, \triangleright_M, \rho^M, \alpha_M)$ ,  $(N, \triangleright_N, \rho^N, \alpha_N)$ ,  $(P, \triangleright_P, \rho^P, \alpha_P) \in {}^H_H \mathbb{YD}$ . With notation as above, define the linear map

$$a_{M,N,P} : (M \otimes N) \otimes P \longrightarrow M \otimes (N \otimes P), \quad (m \otimes n) \otimes p \mapsto \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p)),$$

where  $m \in M$ ,  $n \in N$  and  $p \in P$ . Then  $a_{M,N,P}$  is an isomorphism of left  $(H, \beta)$ -Hom-modules and left  $(H, \beta)$ -Hom-comodules.

**Proof** Same to the proof of [16, Proposition 3.2].  $\square$

**Lemma 4.6** Let  $(H, \beta)$  be a Hom-bialgebra and  $(M, \triangleright_M, \rho^M, \alpha_M)$ ,  $(N, \triangleright_N, \rho^N, \alpha_N) \in {}^H_H \mathbb{YD}$ . Define the linear map

$$c_{M,N} : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes \alpha_M^{-1}(m_0),$$

where  $m \in M$  and  $n \in N$ . Then  $c_{M,N}$  is a morphism of left  $(H, \beta)$ -Hom-modules and left  $(H, \beta)$ -Hom-comodules.

**Proof** For all  $h \in H$ ,  $m \in M$  and  $n \in N$ , firstly,

$$\begin{aligned} & (\alpha_N \otimes \alpha_M) \circ c_{M,N}(m \otimes n) \\ &= \alpha_N(\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes m_0 \\ &\stackrel{(HM1)}{=} (\beta^3(m_{-1}) \triangleright_N n) \otimes m_0 \\ &\stackrel{(HCM1)}{=} (\beta^2(\alpha_M(m)_{-1}) \triangleright_N \alpha_N^{-1}(\alpha_N(n)) \otimes \alpha_M^{-1}(\alpha_M(m)_0) \\ &= c_{M,N} \circ (\alpha_M \otimes \alpha_N)(m \otimes n); \end{aligned}$$

secondly,

$$\begin{aligned} c_{M,N}(h \triangleright_{M \otimes N} (m \otimes n)) &= c_{M,N}((h_1 \triangleright_M m) \otimes (h_2 \triangleright_N n)) \\ &= (\beta^2((h_1 \triangleright_M m)_{-1}) \triangleright_N \alpha_N^{-1}(h_2 \triangleright_N n)) \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0) \\ &\stackrel{(HM1)}{=} (\beta^2((h_1 \triangleright_M m)_{-1}) \triangleright_N (\beta^{-1}(h_2) \triangleright_N \alpha_N^{-1}(n))) \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0) \\ &\stackrel{(HM2)}{=} ((\beta((h_1 \triangleright_M m)_{-1}) \beta^{-1}(h_2)) \triangleright_N n) \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0) \\ &\stackrel{(HA1)}{=} (\beta((h_1 \triangleright_M m)_{-1}) \beta^{-2}(h_2)) \triangleright_N n \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0) \\ &\stackrel{(HYD)}{=} (\beta(\beta^{-2}(h_1) \beta(m_{-1})) \triangleright_N n) \otimes \alpha_M^{-1}(\beta^3(\beta^{-2}(h_2)) \triangleright_M m_0) \\ &\stackrel{(HC1)}{=} (\beta(\beta^{-2}(h_1) \beta(m_{-1})) \triangleright_N n) \otimes \alpha_M^{-1}(\beta^3(\beta^{-2}(h_2)) \triangleright_M m_0) \\ &= ((\beta^{-1}(h_1) \beta^2(m_{-1})) \triangleright_N n) \otimes \alpha_M^{-1}(\beta(h_2) \triangleright_M m_0) \\ &\stackrel{(HM1)}{=} ((\beta^{-1}(h_1) \beta^2(m_{-1})) \triangleright_N n) \otimes (h_2 \triangleright_M \alpha_M^{-1}(m_0)) \\ &\stackrel{(HM2)}{=} (h_1 \triangleright_N (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))) \otimes (h_2 \triangleright_M \alpha_M^{-1}(m_0)) \\ &= h \triangleright_{N \otimes M} ((\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes \alpha_M^{-1}(m_0)) \\ &= h \triangleright_{N \otimes M} c_{M,N}(m \otimes n); \end{aligned}$$

finally,

$$\begin{aligned}
& (\rho^{N \otimes M} \circ c_{M,N})(m \otimes n) \\
&= \beta^{-2}((\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_{-1} \alpha_M^{-1}(m_0)_{-1}) \otimes (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_0 \\
&\quad \otimes \alpha_M^{-1}(m_0)_0 \\
&\stackrel{(HCM1)}{=} \beta^{-2}((\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_{-1} \beta^{-1}(m_{0-1})) \otimes (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_0 \\
&\quad \otimes \alpha_M^{-1}(m_{00}) \\
&\stackrel{(HCM2)}{=} \beta^{-2}((\beta(m_{-11}) \triangleright_N \alpha_N^{-1}(n))_{-1} \beta^{-1}(m_{-12})) \otimes (\beta(m_{-11}) \triangleright_N \alpha_N^{-1}(n))_0 \\
&\quad \otimes m_0 \\
&\stackrel{(HC1)}{=} \beta^{-2}((\beta^2(\beta^{-1}(m_{-1})_1) \triangleright_N \alpha_N^{-1}(n))_{-1} \beta^{-1}(m_{-1})_2) \\
&\quad \otimes (\beta^2(\beta^{-1}(m_{-1})_1) \triangleright_N \alpha_N^{-1}(n))_0 \otimes m_0 \\
&\stackrel{(HYD)}{=} \beta^{-2}(\beta^{-1}(m_{-1})_1 \beta(\alpha_N^{-1}(n)_{-1})) \otimes (\beta^3(\beta^{-1}(m_{-1})_2) \triangleright_N \alpha_N^{-1}(n)_0) \otimes m_0 \\
&\stackrel{(HC1)(HA1)}{=} \beta^{-3}(m_{-11}) \beta^{-1}(\alpha_N^{-1}(n)_{-1}) \otimes (\beta^{-2}(m_{-12}) \triangleright_N \alpha_N^{-1}(n)_0) \otimes m_0 \\
&\stackrel{(HCM1)}{=} \beta^{-3}(m_{-11}) \beta^{-2}(n_{-1}) \otimes (\beta^{-2}(m_{-12}) \triangleright_N \alpha_N^{-1}(n_0)) \otimes m_0 \\
&\stackrel{(HCM2)}{=} \beta^{-2}(m_{-1}) \beta^{-2}(n_{-1}) \otimes (\beta^{-2}(m_{0-1}) \triangleright_N \alpha_N^{-1}(n_0)) \otimes \alpha_M^{-1}(m_{00}) \\
&\stackrel{(HA1)}{=} \beta^{-2}(m_{-1} n_{-1}) \otimes (\beta^{-2}(m_{0-1}) \triangleright_N \alpha_N^{-1}(n_0)) \otimes \alpha_M^{-1}(m_{00}) \\
&= (id \otimes c_{M,N})(\beta^{-2}(m_{-1} n_{-1}) \otimes m_0 \otimes n_0) \\
&= (id \otimes c_{M,N}) \circ \rho^{M \otimes N}(m \otimes n).
\end{aligned}$$

Thus  $c_{M,N}$  is a morphism of left  $(H, \beta)$ -Hom-modules and left  $(H, \beta)$ -Hom-comodules.  $\square$

**Remarks** (1) The pre-braiding  $(c_{M,N})$  differs from the one in [16, Proposition 3.3].

(2) If  $(H, \beta)$  is a Hom-Hopf algebra with bijective antipode  $S$ , then the pre-braiding  $(c_{M,N})$  is invertible with

$$c_{M,N}^{-1} : N \otimes M \rightarrow M \otimes N, n \otimes m \mapsto \alpha_M^{-1}(m_{(0)}) \otimes S^{-1}(\beta^2(m_{(-1)})) \triangleright \alpha_N^{-1}(n).$$

**Theorem 4.7** Let  $(H, \beta)$  is a Hom-Hopf algebra with bijective antipode  $S$ . Then the Hom-Yetter-Drinfeld category  ${}^H_H \mathbb{YD}$  is a braided tensor category, with tensor product, associativity constraints, and braiding defined in Lemmas 4.4, 4.5 and 4.6, respectively, and the unit  $I = (K, id_K)$ .

**Proof** The proof of the pentagon axiom for  $a_{M,N,P}$  is same to the proof of [16, Theorem 3.4]. Next we prove that the hexagonal relation for  $c_{M,N}$ . Let  $(M, \triangleright_M, \rho^M, \alpha_M)$ ,  $(N, \triangleright_N, \rho^N, \alpha_N)$ ,  $(P, \triangleright_P, \rho^P, \alpha_P) \in {}^H_H \mathbb{YD}$ . Then for all  $m \in M$ ,  $n \in N$  and  $p \in P$ , we have

$$\begin{aligned}
& ((id_N \otimes c_{M,P}) \circ (a_{N,M,P}) \circ (c_{M,N} \otimes id_P))((m \otimes n) \otimes p) \\
&= \alpha_N^{-1}(\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes ((\beta^2(\alpha_M^{-1}(m_0)_{-1}) \triangleright_P p) \otimes \alpha_M^{-1}(\alpha_M^{-1}(m_0)_0))
\end{aligned}$$



$$\begin{aligned}
& \stackrel{(HCM1)}{=} \alpha_N^{-1}(\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes ((\beta(m_{0-1}) \triangleright_P p) \otimes \alpha_M^{-2}(m_{00})) \\
& \stackrel{(HCM2)}{=} \alpha_N^{-1}(\beta(m_{-11}) \triangleright_N \alpha_N^{-1}(n)) \otimes ((\beta(m_{-12}) \triangleright_P p) \otimes \alpha_M^{-1}(m_0)) \\
& \stackrel{(HC1)}{=} \alpha_N^{-1}(\beta(m_{-1})_1 \triangleright_N \alpha_N^{-1}(n)) \otimes ((\beta(m_{-1})_2 \triangleright_P p) \otimes \alpha_M^{-1}(m_0)) \\
& \stackrel{(HCM1)}{=} \alpha_N^{-1}(\beta^2(\alpha_M^{-1}(m)_{-1})_1 \triangleright_N \alpha_N^{-1}(n)) \otimes ((\beta^2(\alpha_M^{-1}(m)_{-1})_2 \triangleright_P p) \otimes \alpha_M^{-1}(m)_0) \\
& = (a_{N,P,M} \circ c_{M,N \otimes P} \circ a_{M,N,P})(m \otimes n) \otimes p,
\end{aligned}$$

and

$$\begin{aligned}
& ((c_{M,P} \otimes id_N) \circ (a_{N,M,P}^{-1} \circ (id_M \otimes c_{N,P}))(m \otimes (n \otimes p)) \\
& = ((\beta^2(\alpha_M(m)_{-1}) \triangleright_P \alpha_P^{-1}(\beta^2(n_{-1}) \triangleright_P \alpha_P^{-1}(p))) \otimes \alpha_M^{-1}(\alpha_M(m)_0)) \otimes \alpha_N^{-2}(n_0) \\
& \stackrel{(HM1)}{=} ((\beta^2(\alpha_M(m)_{-1}) \triangleright_P (\beta(n_{-1}) \triangleright_P \alpha_P^{-2}(p))) \otimes \alpha_M^{-1}(\alpha_M(m)_0)) \otimes \alpha_N^{-2}(n_0) \\
& \stackrel{(HM2)}{=} (((\beta(\alpha_M(m)_{-1})\beta(n_{-1})) \triangleright_P \alpha_P^{-1}(p)) \otimes \alpha_M^{-1}(\alpha_M(m)_0)) \otimes \alpha_N^{-2}(n_0) \\
& \stackrel{(HM1)(HA1)}{=} (\alpha_P((\alpha_M(m)_{-1}n_{-1})) \triangleright_P \alpha_P^{-2}(p)) \otimes \alpha_M^{-1}(\alpha_M(m)_0) \otimes \alpha_N^{-2}(n_0) \\
& = (a_{P,M,N}^{-1} \circ c_{M \otimes N,P} \circ a_{M,N,P}^{-1})(m \otimes (n \otimes p)),
\end{aligned}$$

finishing the proof.  $\square$

By Theorem 3.3, 3.6 and 4.7, we can get the main result in this paper.

**Theorem 4.8** Let  $(H, \beta)$  is a Hom-Hopf algebra with bijective antipode  $S$ ,  $(A, \alpha)$  a left  $(H, \beta)$ -module Hom-algebra and a left  $(H, \beta)$ -comodule Hom-coalgebra satisfying  $\beta^2 = id_H$ . Then  $(A_{\diamond}^{\natural}H, \mu_{A_{\natural}H}, 1_A \otimes 1_H, \Delta_{A \circ H}, \varepsilon_A \otimes \varepsilon_H, \alpha \otimes \beta)$  is a Radford biproduct Hom-bialgebra if and only if  $(A, \alpha)$  is a bialgebra in the Hom-Yetter-Drinfeld category  ${}^H_H\mathbb{YD}$ .

**Proof** It is obvious if we compare the conditions (R4), (R5) in Theorem 3.3 and the condition (HYD) in Definition 4.1, the definition of pre-braiding  $c_{M,N}$  in Lemma 4.6, respectively.  $\square$

**Remarks** (1) If  $\alpha = id_A$  and  $\beta = id_H$  in Theorem 4.8, then we can get the Majid's conclusion about the usual Radford biproduct and Yetter-Drinfeld category.

(2)  $(A_{\diamond}^{\natural}H, \mu_{A_{\natural}H}, 1_A \otimes 1_H, \Delta_{A \circ H}, \varepsilon_A \otimes \varepsilon_H, \alpha \otimes \beta, S_{A_{\diamond}^{\natural}H})$  is a Radford biproduct Hom-Hopf algebra if and only if  $(A, \alpha, S_A)$  is a Hopf algebra in the Hom-Yetter-Drinfeld category  ${}^H_H\mathbb{YD}$ .

## 5 Applications

In this section, we give some applications of the above sections.

**Example 5.1** Let  $K\mathbb{Z}_2 = K\{1, a\}$  be Hopf group algebra (see [23]). Then  $(K\mathbb{Z}_2, id_{K\mathbb{Z}_2})$  is a Hom-Hopf algebra.

Let  $T_{2,-1} = K\{1, g, x, y | g^2 = 1, x^2 = 0, y = gx, gy = -gy = x\}$  be Taft's Hopf algebra (see [13]), its coalgebra structure and antipode are given by

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes g + 1 \otimes x, \Delta(y) = y \otimes 1 + g \otimes y;$$

$$\varepsilon(g) = 1, \varepsilon(x) = 0, \varepsilon(y) = 0;$$

and

$$S(g) = g, S(x) = y, S(y) = -x.$$

Define a linear map  $\alpha: T_{2,-1} \longrightarrow T_{2,-1}$  by

$$\alpha(1) = 1, \alpha(g) = g, \alpha(x) = kx, \alpha(y) = ky$$

where  $0 \neq k \in K$ . Then  $\alpha$  is an automorphism of Hopf algebras.

So we can get a Hom-Hopf algebra  $H_\alpha = (T_{2,-1}, \alpha \circ \mu_{T_{2,-1}}, 1_{T_{2,-1}}, \Delta_{T_{2,-1}} \circ \alpha, \varepsilon_{T_{2,-1}}, \alpha)$  (see [19]).

**Lemma 5.1.1** With notations above. Define module action  $\triangleright : K\mathbb{Z}_2 \otimes H_\alpha \longrightarrow H_\alpha$  by

$$1_{K\mathbb{Z}_2} \triangleright 1_{H_\alpha} = 1_{H_\alpha}, 1_{K\mathbb{Z}_2} \triangleright g = g,$$

$$1_{K\mathbb{Z}_2} \triangleright x = kx, 1_{K\mathbb{Z}_2} \triangleright y = ky,$$

$$a \triangleright 1_{H_\alpha} = 1_{H_\alpha}, a \triangleright g = g,$$

$$a \triangleright x = kx, a \triangleright y = ky,$$

Then by a routine computation we can get  $(H_\alpha, \triangleright, \alpha)$  is a  $(K\mathbb{Z}_2, id_{K\mathbb{Z}_2})$ -module Hom-algebra. Therefore,  $(H_\alpha \bowtie K\mathbb{Z}_2, \alpha \otimes id_{K\mathbb{Z}_2})$  is a smash product Hom-algebra.

**Lemma 5.1.2** With notations above. Define comodule action  $\rho : H_\alpha \longrightarrow K\mathbb{Z}_2 \otimes H_\alpha$  by

$$\rho : H_\alpha \longrightarrow K\mathbb{Z}_2 \otimes H_\alpha$$

$$1_{H_\alpha} \mapsto 1_{K\mathbb{Z}_2} \otimes 1_{H_\alpha}$$

$$g \mapsto 1_{K\mathbb{Z}_2} \otimes g$$

$$x \mapsto ka \otimes x$$

$$y \mapsto ka \otimes y.$$

Then we can get  $(H_\alpha, \rho, \alpha)$  is a left  $(K\mathbb{Z}_2, id_{K\mathbb{Z}_2})$ -comodule Hom-coalgebra by a direct computation. Therefore,  $(H_\alpha \bowtie K\mathbb{Z}_2, \alpha \otimes id_{K\mathbb{Z}_2})$  is a smash coproduct Hom-coalgebra.

By the above two lemmas and a direct computation, we have

**Theorem 5.1.3** With notations above.  $(H_{\alpha} \bowtie K\mathbb{Z}_2, \mu_{H_{\alpha} \bowtie K\mathbb{Z}_2}, 1_{H_{\alpha}} \otimes 1_{K\mathbb{Z}_2}, \Delta_{H_{\alpha} \bowtie K\mathbb{Z}_2}, \varepsilon_{H_{\alpha}} \otimes \varepsilon_{K\mathbb{Z}_2}, \alpha \otimes id_{K\mathbb{Z}_2})$  is a Radford biproduct Hom-bialgebra. Furthermore,  $(H_{\alpha} \bowtie K\mathbb{Z}_2, \alpha \otimes id_{K\mathbb{Z}_2}, S_{H_{\alpha} \bowtie K\mathbb{Z}_2})$  is a Hom-Hopf algebra, where  $S_{H_{\alpha} \bowtie K\mathbb{Z}_2}$  is defined by

$$\begin{aligned} S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(1_{H_{\alpha}} \otimes 1_{K\mathbb{Z}_2}) &= 1_{H_{\alpha}} \otimes 1_{K\mathbb{Z}_2}; & S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(1_{H_{\alpha}} \otimes a) &= 1_{H_{\alpha}} \otimes a \\ S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(g \otimes 1_{K\mathbb{Z}_2}) &= g \otimes 1_{K\mathbb{Z}_2}; & S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(g \otimes a) &= g \otimes a \\ S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(x \otimes 1_{K\mathbb{Z}_2}) &= y \otimes a; & S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(x \otimes a) &= y \otimes 1_{K\mathbb{Z}_2} \\ S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(y \otimes 1_{K\mathbb{Z}_2}) &= -x \otimes a; & S_{H_{\alpha} \bowtie K\mathbb{Z}_2}(y \otimes a) &= -x \otimes 1_{K\mathbb{Z}_2}. \end{aligned}$$

**Example 5.2** Let  $K\mathbb{Z}_2 = K\{1, a\}$  be Hopf group algebra (see [23]). Then  $(K\mathbb{Z}_2, id_{K\mathbb{Z}_2})$  is a Hom-Hopf algebra.

Let  $A = K\{1, z\}$  be a vector space. Define the multiplication  $\mu_A$  by

$$1z = z1 = lz, \quad z^2 = 0$$

and the automorphism  $\beta : A \longrightarrow A$  by

$$\beta(1) = 1, \quad \beta(z) = lz$$

where  $0 \neq l \in K$ . Then  $(A, \beta)$  is a Hom-algebra.

Define the comultiplication  $\Delta_A$  by

$$\Delta_A(1) = 1 \otimes 1, \quad \Delta_A(z) = lz \otimes 1 + l1 \otimes z, \quad \text{and} \quad \varepsilon_A(1) = 1, \quad \varepsilon_A(z) = 0.$$

Then  $(A, \beta)$  is a Hom-coalgebra.

**Lemma 5.2.1** With notations above. Define module action  $\triangleright : K\mathbb{Z}_2 \otimes A \longrightarrow A$  by

$$1_{K\mathbb{Z}_2} \triangleright 1_A = 1_A, \quad 1_{K\mathbb{Z}_2} \triangleright z = lz,$$

$$a \triangleright 1_A = 1_A, \quad a \triangleright z = -lz,$$

Then by a routine computation we can get  $(A, \triangleright, \beta)$  is a  $(K\mathbb{Z}_2, id_{K\mathbb{Z}_2})$ -module Hom-algebra. Therefore,  $(A \bowtie K\mathbb{Z}_2, \beta \otimes id_{K\mathbb{Z}_2})$  is a smash product Hom-algebra.

**Lemma 5.2.2** With notations above. Define comodule action  $\psi : A \longrightarrow K\mathbb{Z}_2 \otimes A$  by

$$\begin{aligned} \psi : A &\longrightarrow K\mathbb{Z}_2 \otimes A \\ 1_A &\mapsto 1_{K\mathbb{Z}_2} \otimes 1_A \\ z &\mapsto la \otimes z. \end{aligned}$$

Then we can get  $(A, \psi, \beta)$  is a left  $(K\mathbb{Z}_2, id_{K\mathbb{Z}_2})$ -comodule Hom-coalgebra by a direct computation. Therefore,  $(A \bowtie K\mathbb{Z}_2, \beta \otimes id_{K\mathbb{Z}_2})$  is a smash coproduct Hom-coalgebra.

By the above two lemmas and a direct computation, we have

**Theorem 5.2.3** With notations above.  $(A_{\diamond}^{\natural}K\mathbb{Z}_2, \mu_{A_{\natural}K\mathbb{Z}_2}, 1_A \otimes 1_{K\mathbb{Z}_2}, \Delta_{A_{\diamond}K\mathbb{Z}_2}, \varepsilon_A \otimes \varepsilon_{K\mathbb{Z}_2}, \beta \otimes id_{K\mathbb{Z}_2})$  is a Radford biproduct Hom-bialgebra. Furthermore,  $(A_{\diamond}^{\natural}K\mathbb{Z}_2, \beta \otimes id_{K\mathbb{Z}_2}, S_{A_{\diamond}^{\natural}K\mathbb{Z}_2})$  is a Hom-Hopf algebra, where  $S_{A_{\diamond}^{\natural}K\mathbb{Z}_2}$  is defined by

$$\begin{aligned} S_{A_{\diamond}^{\natural}K\mathbb{Z}_2}(1_A \otimes 1_{K\mathbb{Z}_2}) &= 1_A \otimes 1_{K\mathbb{Z}_2}; & S_{A_{\diamond}^{\natural}K\mathbb{Z}_2}(1_A \otimes a) &= 1_A \otimes a \\ S_{A_{\diamond}^{\natural}K\mathbb{Z}_2}(z \otimes 1_{K\mathbb{Z}_2}) &= z \otimes a; & S_{A_{\diamond}^{\natural}K\mathbb{Z}_2}(z \otimes a) &= -z \otimes 1_{K\mathbb{Z}_2}. \end{aligned}$$

**Remark** If  $\beta = id_A$ , i.e.,  $l = 1$ , then Example 5.2 is same to the biproduct  $B \star H$  (which is isomorphic to the Sweedler's Hopf algebra  $T_{2,\omega}$ ) in [12, Example 4.3].

In the following, let us recall the definition of quasitriangular Hom-Hopf algebra in [26] or [10].

A quasitriangular Hom-Hopf algebra is a octuple  $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta, R)$  (abbr.  $(H, \beta, R)$ ) in which  $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta)$  is a Hom-Hopf algebra and  $R = R^1 \otimes R^2 \in H \otimes H$ , satisfying the following axioms (for all  $h \in H$  and  $R=r$ ):

$$\begin{aligned} (QHA1) \quad & \varepsilon(R^1)R^2 = R^1\varepsilon(R^2) = 1, \\ (QHA2) \quad & R^1_1 \otimes R^1_2 \otimes \beta(R^2) = \beta(R^1) \otimes \beta(r^1) \otimes R^2r^2, \\ (QHA3) \quad & \beta(R^1) \otimes R^2_1 \otimes R^2_2 = R^1r^1 \otimes \beta(r^2) \otimes \beta(R^2), \\ (QHA4) \quad & h_2R^1 \otimes h_1R^2 = R^1h_1 \otimes R^2h_2, \\ (QHA5) \quad & \beta(R^1) \otimes \beta(R^2) = R^1 \otimes R^2. \end{aligned}$$

Let  $(H, \beta, S)$  be a Hom-Hopf algebra and  $R = R^1 \otimes R^2 \in H \otimes H$ . Define:

$$\rho^H : H \longrightarrow H \otimes H \quad h \mapsto h_{-1} \otimes h_0 = \beta^{-3}(R^2) \otimes R^1h.$$

**Proposition 5.3** Let  $(H, \beta, R)$  be a quasitriangular Hom-Hopf algebra. Then  $(H, \beta, \rho^H)$  is a left  $(H, \beta)$ -comodule Hom-coalgebra and  $(H, \mu_H, \rho^H, \beta)$  is a Hom-Yetter-Drinfeld module.

**Proof** We compute as follows:

$$\begin{aligned} \beta(h_{-1}) \otimes \beta(h_0) &= \beta(\beta^{-3}(R^2)) \otimes \beta(R^1h) \\ &\stackrel{(HA1)}{=} \beta(\beta^{-3}(R^2)) \otimes \beta(R^1)\beta(h) \\ &\stackrel{(QHA5)}{=} \beta^{-3}(R^2) \otimes R^1\beta(h) = \beta(h)_{-1} \otimes \beta(h)_0, \end{aligned}$$

so  $(HCM1)$  holds.

$$\begin{aligned} h_{-11} \otimes h_{-12}\beta(h_0) &= \beta^{-3}(R^2)_1 \otimes \beta^{-3}(R^2)_2 \otimes \beta(R^1h) \\ &\stackrel{(HC1)(HA1)}{=} \beta^{-3}(R^2_1) \otimes \beta^{-3}(R^2_2) \otimes \beta(R^1)\beta(h) \\ &\stackrel{(QHA3)}{=} \beta^{-2}(R^2) \otimes \beta^{-2}(r^2) \otimes (r^1R^1)\beta(h) \end{aligned}$$

$$\begin{aligned}
& \stackrel{(HA2)}{=} \beta^{-2}(R^2) \otimes \beta^{-2}(r^2) \otimes \beta(r^1)(R^1 h) \\
& \stackrel{(QHA5)}{=} \beta^{-2}(R^2) \otimes \beta^{-3}(r^2) \otimes r^1(R^1 h) \\
& = \beta(h_{-1}) \otimes h_{0-1} \otimes h_{00},
\end{aligned}$$

thus we get (HCM2).

$$\begin{aligned}
\beta^2(h_{-1}) \otimes h_{01} \otimes h_{02} & = \beta^{-1}(R^2) \otimes (R^1 h)_1 \otimes (R^1 h)_2 \\
& = \beta^{-1}(R^2) \otimes R^1_1 h_1 \otimes R^1_2 h_2 \\
& \stackrel{(QHA2)}{=} \beta^{-2}(R^2 r^2) \otimes \beta(R^1) h_1 \otimes \beta(r^1) h_2 \\
& \stackrel{(QHA5)(HA1)}{=} \beta^{-3}(R^2) \beta^{-3}(r^2) \otimes R^1 h_1 \otimes r^1 h_2 \\
& = h_{1-1} h_{1-1} \otimes h_{10} \otimes h_{20},
\end{aligned}$$

therefore we obtain (HCMC1).

(HCMC2) can be checked by (QHA1).

Finally we verify that (HYD) is satisfied.

$$\begin{aligned}
(\beta^2(h_1) \triangleright g)_{-1} h_2 \otimes (\beta^2(h_1) \triangleright g)_0 & = \beta^{-3}(R^2) h_2 \otimes R^1(\beta^2(h_1) g) \\
& \stackrel{(HA2)}{=} \beta^{-3}(R^2) h_2 \otimes (\beta^{-1}(R^1) \beta^2(h_1)) \beta(g) \\
& \stackrel{(HA1)(HC1)}{=} \beta^{-3}(R^2 \beta^3(h)_2) \otimes \beta^{-1}(R^1 \beta^3(h)_1) \beta(g) \\
& \stackrel{(QHA4)}{=} \beta^{-3}(\beta^3(h)_1 R^2) \otimes \beta^{-1}(\beta^3(h)_2 R^1) \beta(g) \\
& \stackrel{(HA1)(HC1)}{=} h_1 \beta^{-3}(R^2) \otimes (\beta^2(h_2) \beta^{-1}(R^1)) \beta(g) \\
& \stackrel{(HA2)}{=} h_1 \beta^{-3}(R^2) \otimes \beta^3(h_2) (\beta^{-1}(R^1) g) \\
& \stackrel{(QHA5)}{=} h_1 \beta^{-2}(R^2) \otimes \beta^3(h_2) (R^1 g) \\
& = h_1 \beta(g_{-1}) \otimes (\beta^3(h_2) \triangleright g_0),
\end{aligned}$$

finishing the proof.  $\square$

**Proposition 5.4** Let  $(H, \beta, S)$  be a Hom-Hopf algebra, with notations as above. If  $(H, \beta, \rho^H)$  is a left  $(H, \beta)$ -comodule Hom-coalgebra and  $(H, \mu_H, \rho^H, \beta)$  is a Hom-Yetter-Drinfeld module, then  $(H, \beta, R)$  is a quasitriangular Hom-Hopf algebra.

**Proof** It is straightforward.  $\square$

By Proposition 5.3 and 5.4, we have:

**Theorem 5.5** With notations as above.  $(H, \beta, R)$  is a quasitriangular Hom-Hopf algebra if and only if  $(H, \beta, \rho^H)$  is a left  $(H, \beta)$ -comodule Hom-coalgebra and  $(H, \mu_H, \rho^H, \beta)$  is a Hom-Yetter-Drinfeld module.

Dually, we have

**Theorem 5.6** Let  $(H, \beta, S)$  be a Hom-Hopf algebra and  $\sigma : H \otimes H \longrightarrow K$  a bilinear map. Define  $\triangleright_H : H \otimes H \longrightarrow H$  by

$$h \otimes g \mapsto h \triangleright_H g = \sigma(g_1, \beta^{-3}(h))g_2,$$

where  $h, g \in H$ . Then  $(H, \beta, \sigma)$  is a cobraided Hom-Hopf algebra (see [11, 28]) if and only if  $(H, \beta, \triangleright_H)$  is a left  $(H, \beta)$ -module Hom-algebra and  $(H, \triangleright_H, \Delta_H, \beta)$  is a Hom-Yetter-Drinfeld module.

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